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# The role of orthogonal polynomials in the six-vertex model and its combinatorial applications*

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**Abstract**

The Hankel determinant representations for the partition function and boundary correlation functions of the six-vertex model with domain wall boundary conditions are investigated by the methods of orthogonal polynomial theory. For specific values of the parameters of the model, corresponding to 1-, 2- and 3-enumerations of Alternating Sign Matrices (ASMs), these polynomials specialize to classical ones (Continuous Hahn, Meixner-Pollaczek, and Continuous Dual Hahn, respectively). As a consequence, a unified and simplified treatment of ASMs enumerations turns out to be possible, leading also to some new results such as the refined 3-enumerations of ASMs. Furthermore, the use of orthogonal polynomials allows us to express, for generic values of the parameters of the model, the partition function of the (partially) *inhomogeneous* model in terms of the one-point boundary correlation functions of the *homogeneous* one.

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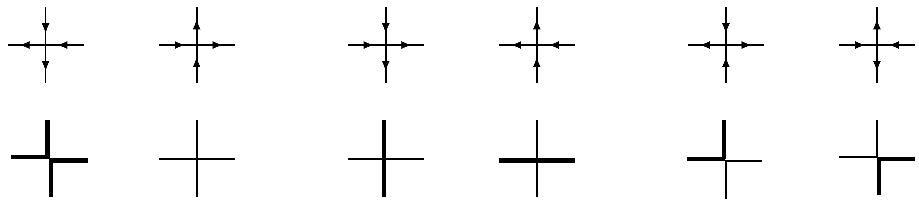
# 1 Introduction

We shall consider here the six-vertex model on a square lattice with domain wall boundary conditions (DWBC). The model, in its inhomogeneous formulation, i.e., with position dependent Boltzmann weights, was originally proposed in [1], within the theory of correlation functions of quantum integrable models, in the framework of the quantum inverse scattering method [2]. It was subsequently solved in [3], where a determinant formula for the partition function was obtained and proven (see also [4]). Analogous determinant formulae have been given also for the one- and two-point boundary correlation functions [5, 6]. In its homogeneous version, the six-vertex model with DWBC admits usual interpretation as a model of statistical mechanics with fixed boundary conditions, and it may be seen as a variation of the original model with periodic boundary conditions [7, 8]. The model is known to be closely related with the problems of enumeration of alternating sign matrices (ASMs) and domino tilings (see book [9] for a nice review). It should be mentioned that ASM enumerations in turn emerge, via Razumov-Stroganov conjecture [10, 11], in the context of some quantum spin chains and loop models; for recent works, see for instance [12–14] and references therein.

Till now, specific results for the six-vertex model with DWBC at particular values of its parameters (obtained mainly in application to ASM enumerations) were derived from general results for the inhomogeneous version, first specializing the parameters to the considered case, and next performing the homogeneous limit. Each time, the homogeneous limit was an hard task on its own right, and a specific approach was devised to work it out in each single case [15–17]. More recently, especially in the context of Razumov-Stroganov conjecture, where the model is considered at the so-called ice-point, the homogeneous limit of the partition function is often performed only on a subset of the spectral parameters, the remaining ones being reinterpreted as the variables of some generating function encoding some peculiar properties of the model [17, 18].

Here we consider a different approach, which turns out to be rather convenient: we start directly from the homogeneous limit representations, worked out once for all for the model with generic vertex weights. Such 'Hankel determinant' representations has been derived in [4] for the partition function, and in [5] and [6], respectively, for the one- and two-point boundary correlators. Following a rather standard procedure, these quantities can then be expressed in terms of orthogonal polynomials. For specific values of the vertex weights corresponding to the so-called 'ice-point', 'free-fermion line' and 'dual ice-point' of the six-vertex model, these orthogonal polynomials specialize to classical ones (i.e., of hypergeometric type, or, equivalently, belonging to the Askey scheme, see for example [19]), allowing the evaluation of the partition function and boundary correlators in closed form. These three cases correspond to the 1-, 2- and 3-enumerations of ASMs, respectively; a unified and simplified treatment of known ASMs is thus provided, leading also to some new results such as the refined 3-enumerations of ASMs.

For generic values of the vertex weights, such orthogonal polynomial representations allow to express the two-point boundary correlator in terms of the analogous one-point



$w_1$ $w_2$ $w_3$ $w_4$ $w_5$ $w_6$

Figure 1: The six allowed types of vertices in terms of arrows (first row), in terms of lines (second row), and their Boltzmann weights (third row).

boundary correlator. Such relationship can in turn be understood as follows: the partition function of the model with two inhomogeneities may be written in terms of the one point boundary correlator solely. This simple result can be extended further: indeed here we show that it is possible to 'get away' from the homogeneous limit and recover the structure of the inhomogeneous partition function, thus generalizing previous formulae of Stroganov [17] to a generic number of unfixed spectral parameters. Specifically, we provide a simple expression for the partition function of the (partially) *inhomogeneous* model in terms of the one-point boundary correlators of the *homogeneous* model; our result is valid for any choice of the crossing parameter. This somehow simplifies and render explicit previous results discussed in [18], where the partition function of the inhomogeneous model (at ice-point) was expressed in terms of Shur functions.

# 2 The six-vertex model with DWBC

The six-vertex model, which was originally proposed as a model of two-dimensional ice (hence the alternative denomination: 'square ice'), is formulated on a square lattice with arrows lying on edges, and obeying the so-called 'ice-rule', namely, the only admitted configurations are such that there are always two arrows pointing away from, and two arrows pointing into, each lattice vertex. An equivalent and graphically simpler description of the configurations of the model can be given in terms of lines flowing through the vertices: for each arrow pointing downward or to the left, draw a thick line on the corresponding link. The six possible vertex states and the Boltzmann weights $w_i$ assigned to each vertex according to its state $i$ $(i = 1, \dots, 6)$ are shown in figure 1. We shall presently restrict ourselves to the homogeneous version of the model, where the Boltzmann weights are site independent. We shall however turn to the inhomogeneous version of the model in the last section.

The DWBC are imposed on the $N \times N$ square lattice by fixing the direction of all arrows on the boundaries in a specific way. Namely, the vertical arrows on the top and bottom of the lattice point inward, while the horizontal arrows on the left and right sides point outward. Equivalently, a generic configuration of the model with DWBC can be depicted by $N$ lines flowing from the upper boundary to the left one. This line picture

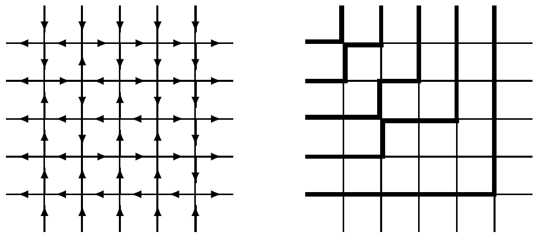



Figure 2: One of the possible configurations of the model with DWBC, in the case$N = 5$: (a) in terms of arrows; (b) in terms of lines.

(besides taking into account the 'ice-rule' in an automated way) is intuitively closer to ASMs recalled in the next section. A possible state of the model both in terms of arrows and of lines is shown in figure 2.

The partition function is defined, as usual, as a sum over all possible arrow configurations, compatible with the imposed DWBC, each configuration being assigned its Boltzmann weight, given as the product of all the corresponding vertex weights,

$$Z_N = \sum_{\substack{\text{arrow configurations}\\ \text{with DWBC}}} \prod_{i=1}^{6} w_i^{n_i} \,. \tag{2.1}$$

Here $n_i$ denotes the number of vertices in the state $i$ in each arrow configuration ($n_1 + \cdots + n_6 = N^2$).

The six-vertex model with DWBC can be considered, with no loss of generality, with its weights invariant under the simultaneous reversal of all arrows,

$$w_1 = w_2 =: a, \qquad w_3 = w_4 =: b, \qquad w_5 = w_6 =: c. \tag{2.2}$$

Under different choices of Boltzmann weights the six-vertex model exhibits different behaviours, according to the value of the parameter $\Delta$, defined as

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}. \tag{2.3}$$

It is well known that there are three physical regions or phases for the six-vertex model: the ferroelectric phase, $\Delta > 1$; the anti-ferroelectric phase, $\Delta < -1$; and, the disordered phase, $-1 < \Delta < 1$. Here we restrict ourselves to the disordered phase, where the Boltzmann weights are conveniently parameterized as

$$a = \sin(\lambda + \eta), \qquad b = \sin(\lambda - \eta), \qquad c = \sin 2\eta. \tag{2.4}$$

With this choice one has $\Delta = \cos 2\eta$. The parameter $\lambda$ is the so-called spectral parameter and $\eta$ is the crossing parameter. The physical requirement of positive Boltzmann weights,

in the disordered regime, restricts the values of the crossing and spectral parameters to $0 < \eta < \pi/2$ and $\eta < \lambda < \pi - \eta$.

An exact representation for the partition function (for generic weights, even complex) was obtained in [4]. When the weights are parameterized according to (2.4) such representation reads

$$Z_N = \frac{[\sin(\lambda-\eta)\sin(\lambda+\eta)]^{N^2}}{\prod_{n=1}^{N-1}(n!)^2} \det_N \Phi \tag{2.5}$$

where $\Phi$ is an $N \times N$ matrix with entries

$$\Phi_{jk} = \partial_\lambda^{j+k}\varphi(\lambda,\eta), \qquad \varphi(\lambda,\eta) = \frac{\sin(2\eta)}{\sin(\lambda-\eta)\sin(\lambda+\eta)}. \tag{2.6}$$

Here and in the following we use the convention that indices of $N \times N$ matrices run over the values $j,k = 0,\dots,N-1$.

This formula for the partition function has been obtained as the homogeneous limit of a more general formula for the inhomogeneous six-vertex model with DWBC. The inhomogeneous model, with site-dependent weights, is defined by introducing two sets of spectral parameters $\{\lambda_\alpha\}_{\alpha=1}^N$ and $\{\nu_\beta\}_{\beta=1}^N$, such that the weights of the vertex lying at the intersection of the $\alpha$-th column with the $\beta$-th row depends on $\lambda_\alpha - \nu_\beta$ rather than simply on $\lambda$, still through formulae (2.4). The inhomogeneous model can be fruitfully investigated through the Quantum Inverse Scattering Method, see papers [1, 3, 4, 20] and book [2] for details. As a result, the partition function of the inhomogeneous model is represented in terms of certain determinant formula which, however, requires some effort in the study of its homogeneous limit, $\nu_\beta \to 0$ and $\lambda_\alpha \to \lambda$, since in this limit the determinant possesses $N^2 - N$ zeros that are cancelled by the same number of singularities coming from the pre-factor. A recipe for taking such a limit was explained in [4] where formula (2.5) was originally obtained. Subsequently, formula (2.5) was used in papers [21, 22] to investigate the thermodynamic limit, $N \to \infty$, of the partition function. In these studies the Hankel structure of the determinant appearing in (2.5), a natural outcome of the homogeneous limit procedure, was exploited through its relation with the Toda chain differential equation and with matrix models.

In addition to the partition function, in the following we shall discuss one- and two-point boundary correlation functions as well. In general, two kinds of one-point correlation functions can be considered in the six-vertex model: the first one ('polarization') is the probability to find an arrow on a given edge in a particular state, while the second one is the probability to find a given vertex in some state $i$. If one restricts to edges or vertices adjacent to the boundary, then such correlation functions are called boundary correlation functions. Following the notations of paper [5], where these boundary correlation functions were studied, let $G_N^{(r)}$ denote the probability that an arrow on the first row and between the $r$-th and $(r+1)$-th columns (enumerated from the right) points left (or, in the line language, that there is a thick line on this edge), and let $H_N^{(r)}$ denote the probability that the first vertex in the $r$-th column (counted from the right) is in the state $i = 5$ (or that

the thick line flows from the top to the left), see Figs. 1 and 2. The first correlation function, $G_N^{(r)}$, is, in fact, the boundary polarization, whose interpretation is more direct from a physical point of view, while the second one, $H_N^{(r)}$, is closely related to the refined enumerations of ASMs. It is easy to see that, due to DWBC, the two correlation functions are related to each other as follows

$$G_N^{(r)} = H_N^{(r)} + H_N^{(r-1)} + \cdots + H_N^{(1)}. \tag{2.7}$$

In [5] both correlation functions were computed using Quantum Inverse Scattering Method for the inhomogeneous six-vertex model. In the homogeneous limit, which is the situation we are interested in here, determinant formulae generalizing (2.5) were found for these correlation functions. For instance, for $H_N^{(r)}$, the following expression was derived

$$H_N^{(r)} = \frac{(N-1)!\,\sin(2\eta)}{\big[\sin(\lambda+\eta)\big]^r\big[\sin(\lambda-\eta)\big]^{N-r+1}}\,\frac{\det_N\Psi}{\det_N\Phi} \tag{2.8}$$

where the matrix $\Psi$ differs from the matrix $\Phi$, equation (2.6), just by the elements of the last column

$$\Psi_{j,k} = \Phi_{j,k}, \quad k = 0,\dots,N-2; \qquad \Psi_{j,N} = \partial_\varepsilon^j \frac{(\sin\varepsilon)^{N-r}[\sin(\varepsilon-2\eta)]^{r-1}}{[\sin(\varepsilon+\lambda-\eta)]^{N-1}}\bigg|_{\varepsilon=0}. \tag{2.9}$$

A similar expression is valid for $G_N^{(r)}$ as well. In what follows we shall focus on $H_N^{(r)}$; the results for $G_N^{(r)}$ will follow immediately from relation (2.7). From the DWBC it immediately follows that $G_N^{(N)} = 1$, and therefore, from (2.7), correlation function $H_N^{(r)}$ satisfies

$$\sum_{r=1}^{N} H_N^{(r)} = 1\,. \tag{2.10}$$

In the following this normalization condition will be used in application to the generating function of $H_N^{(r)}$, defined as

$$H_N(u) := \sum_{r=1}^{N} H_N^{(N-r+1)} u^{r-1}. \tag{2.11}$$

Let us finally recall the two-point boundary correlation function. Again several different definitions are sensible. We shall focus here, for the sake of simplicity and definiteness, on a specific two-point generalization of the one-point correlation function $H_N^{(r)}$, describing the probability of finding vertices of type $i = 5$ on the opposite, top and bottom, boundaries. More precisely, we define $H_N^{(r_1,r_2)}$ as the probability of finding vertices of type $i = 5$ both at the $r_1$-th position of the first row and at the $r_2$-th position of the last row ($r_1$ and $r_2$ are counted from the right).

In [6] such correlation function was computed using Quantum Inverse Scattering Method for the inhomogeneous six-vertex model. In the homogeneous limit, which is the situation we are interested in here, the following determinant formula was derived:

$$
\begin{aligned}
H_N^{(r_1,r_2)} = {}& \frac{(N-1)!\,(N-2)!\,\sin^2(2\eta)}{\big[\sin(\lambda+\eta)\big]^{N+r_1-r_2+1}\big[\sin(\lambda-\eta)\big]^{N+r_2-r_1+1}\det_N\Phi} \\
& \times \left[\det_N\Big(\Phi_{j,k}\Big|\partial_{\varepsilon_2}^j\Big|\partial_{\varepsilon_1}^j\Big)_{0\le j\le N-1, 0\le k\le N-3}\, h_N^{(r_1,r_2)}(\varepsilon_1,\varepsilon_2)\right]\bigg|_{\varepsilon_1=\varepsilon_2=0}. \qquad (2.12)
\end{aligned}
$$

Here the function $h_N^{(r_1,r_2)}(\varepsilon_1,\varepsilon_2)$ is defined as follows:

$$
h_N^{(r_1,r_2)}(\varepsilon_1,\varepsilon_2) = \frac{(\sin\varepsilon_1)^{N-r_1}[\sin(\varepsilon_1-2\eta)]^{r_1-1}(\sin\varepsilon_2)^{N-r_2}[\sin(\varepsilon_2+2\eta)]^{r_2-1}}{\sin(\varepsilon_2-\varepsilon_1+2\eta)[\sin(\varepsilon_1+\lambda-\eta)]^{N-2}[\sin(\varepsilon_2+\lambda+\eta)]^{N-2}}. \qquad (2.13)
$$

The matrix $\Phi$ is defined in (2.6).

It is obvious from the definitions of $H_N^{(r)}$ and $H_N^{(r_1,r_2)}$, that

$$
\sum_{r_1=1}^{N} H_N^{(r_1,r)} = \sum_{r_2=1}^{N} H_N^{(r,r_2)} = H_N^{(r)}\,, \qquad \sum_{r_1,r_2=1}^{N} H_N^{(r_1,r_2)} = 1\,. \qquad (2.14)
$$

In dealing with the two-point boundary function it will be convenient to use the corresponding generating function

$$
H_N(u,v) := \sum_{r=1,s=1}^{N} H_N^{(N-r+1,s)} u^{r-1} v^{s-1}. \qquad (2.15)
$$

Note that (2.14) simply implies $H_N(1,v) = H_N(v,1) = H_N(v)$ where $H_N(v)$ is defined by (2.11); we also have $H_N(1,1) = 1$.

# 3 Orthogonal polynomial representation

## 3.1 Preliminaries

In this section the previously reviewed determinant representations for the partition function and the one- and two-point boundary correlation functions will be analysed by making use of the orthogonal polynomial theory, along the lines proposed in paper [6, 23]. We start with recalling some very standard and well-known facts from the general theory.

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a set of polynomials, with non-vanishing leading coefficient

$$
P_n(x) = \kappa_n x^n + \dots, \qquad \kappa_n \neq 0, \qquad (3.1)
$$

and orthogonal on the real axis with respect to some weight $\mu(x)$,

$$
\int_{-\infty}^{\infty} P_n(x)\, P_m(x)\, \mu(x)\, \mathrm{d}x = h_n \delta_{nm}\,. \qquad (3.2)
$$

Let $c_n$ denote $n$-th moment of the weight $\mu(x)$, i.e.

$$c_n = \int_{-\infty}^{\infty} x^n \mu(x) \mathrm{d}x, \qquad n = 0, 1, \ldots \tag{3.3}$$

and let us consider the $(n+1) \times (n+1)$ determinant

$$D_n = \begin{vmatrix} c_0 & c_1 & \ldots & c_n \\ c_1 & c_2 & \ldots & c_{n+1} \\ \multicolumn{4}{c}{\dotfill} \\ c_n & c_{n+1} & \ldots & c_{2n} \end{vmatrix}. \tag{3.4}$$

Using the orthogonality condition (3.2) and well-known properties of determinants, one can easily derive the following formula

$$D_n = \prod_{k=0}^{n} \frac{h_k}{\kappa_k^2}. \tag{3.5}$$

This formula can be used for computation of Hankel determinants, provided the orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ are known. On the other hand, the polynomials $\{P_n(x)\}_{n=0}^{\infty}$ can in turn be expressed as determinants. For later use let us introduce the notation

$$D_n^{(k)}(x_1, \ldots, x_k) = \begin{vmatrix} c_0 & c_1 & \ldots & c_{n-k} & 1 & 1 & \ldots & 1 \\ c_1 & c_2 & \ldots & c_{n-k+1} & x_1 & x_2 & \ldots & x_k \\ \multicolumn{8}{c}{\dotfill} \\ c_n & c_{n+1} & \ldots & c_{2n-k} & x_1^n & x_2^n & \ldots & x_k^n \end{vmatrix} \tag{3.6}$$

so that $D_n^{(0)} = D_n$, and $D_{n-1}^{(n)}(x_1, \ldots, x_n) = \mathit{\Delta}(x_1, \ldots, x_n)$, where $\mathit{\Delta}(x_1, \ldots, x_n)$ denotes the Vandermonde determinant of $n$ variables. For the polynomials one can find that

$$P_n(x) = \frac{\kappa_n}{D_{n-1}} D_n^{(1)}(x). \tag{3.7}$$

For a proof, see for example, [29].

Relation (3.7) can be read off inversely thus giving an expression for the determinant $D_n^{(1)}(x)$ in terms of the polynomials $P_n(x)$. Taking into account that (see (3.5))

$$\frac{h_n}{\kappa_n^2} = \frac{D_n}{D_{n-1}} \tag{3.8}$$

we can write

$$\frac{D_n^{(1)}(x)}{D_n} = \frac{\kappa_n}{h_n} P_n(x). \tag{3.9}$$

Consider now the case of $D_n^{(2)}(x_1, x_2)$. It is clear that the term of the highest powers on both $x_1$ and $x_2$ is just $D_{n-2}(x_2^n x_1^{n-1} - x_1^n x_2^{n-1})$; following [29], it can be shown that

$$D_n^{(2)}(x_1, x_2) = \frac{D_{n-2}}{\kappa_n \kappa_{n-1}} \big[ P_{n-1}(x_1) P_n(x_2) - P_n(x_1) P_{n-1}(x_2) \big]. \tag{3.10}$$

Again using (3.8), we write

$$\begin{aligned} \frac{D_n^{(2)}(x_1,x_2)}{D_n} &= \frac{\kappa_n\kappa_{n-1}}{h_n h_{n-1}}\,\left[P_{n-1}(x_1)P_n(x_2)-P_n(x_1)P_{n-1}(x_2)\right] \\ &= \frac{\kappa_n\kappa_{n-1}}{h_n h_{n-1}} \begin{vmatrix} P_{n-1}(x_1) & P_{n-1}(x_2) \\ P_n(x_1) & P_n(x_2) \end{vmatrix}. \end{aligned} \tag{3.11}$$

This formula can be easily extended to the general case of $D_n^{(k)}(x_1,\dots,x_k)$. We shall now use all these formulae to provide completely general 'orthogonal polynomial representations' for the partition function and the one- and two-point boundary correlation functions.

## 3.2 The partition function

First we note, following papers [22, 23], that the matrix $\varPhi$ entering the expressions for the homogenous model partition function and the boundary correlation functions can be related with orthogonal polynomials using the integral representation

$$\frac{\sin(2\eta)}{\sin(\lambda-\eta)\sin(\lambda+\eta)} = \int_{-\infty}^{\infty} \mathrm{e}^{x(\lambda-\pi/2)}\frac{\sinh(\eta x)}{\sinh(\pi x/2)}\,\mathrm{d}x. \tag{3.12}$$

This formula is valid if $0<\eta<\pi/2$ and $\eta<\lambda<\pi-\eta$; these values of $\lambda$ and $\eta$ correspond to the so-called disordered regime of the six-vertex model. When considering other regime of the model, the measure of the corresponding polynomials becomes discrete, see [22], but the present procedure may nevertheless be considered, modulo trivial modifications. Indeed, it can be easily seen that our results below do not depend on the particular choice of the regime, and can be extended to other regimes simply using the proper analytical continuation in the parameters $\lambda$ and $\eta$.

Formula (3.12) implies that we have to deal with the set of polynomials which are orthogonal with respect to the following weight function

$$\mu(x) = \mu(x;\lambda,\eta) = \mathrm{e}^{x(\lambda-\pi/2)}\frac{\sinh(\eta x)}{\sinh(\pi x/2)}. \tag{3.13}$$

The corresponding polynomials $P_n(x)=P_n(x;\lambda,\eta)$ also depend on $\lambda$ and $\eta$ which are to be considered as parameters. In what follows we shall often omit the dependence on $\lambda$ and $\eta$ where possible. Let us mention the following useful property of these polynomials

$$P_n(x;\lambda,\eta) = (-1)^n\,P_n(-x;\pi-\lambda,\eta). \tag{3.14}$$

This property can be easily established in virtue of formula (3.7). It is to be mentioned also that both the leading coefficient $\kappa_n=\kappa_n(\lambda,\eta)$ and the normalization constant $h_n = h_n(\lambda,\eta)$ are invariant under the substitution $\lambda\to\pi-\lambda$. The transformation $\lambda\to\pi-\lambda$ is related to the crossing symmetry of the six-vertex model, and the previous property will

have useful consequences in the discussion of the one- and two-point boundary correlation functions, discussed below.

Let us focus now on the partition function. The differential operator in the Hankel matrix entering representation (2.5) pulls down powers of the integration variable, and the matrix itself is immediately rewritten as the matrix of moments of the measure (3.13). Due to (3.5), we readily get

$$Z_N = \left(\frac{\sin 2\eta}{\varphi}\right)^{N^2} \prod_{n=0}^{N-1} \frac{h_n}{(n!)^2 \kappa_n^2} \tag{3.15}$$

where $\varphi = \varphi(\lambda, \eta)$ is exactly the function defining entries of the matrix $\Phi$, see (2.6).

We just want to conclude with a simple remark: whenever the values of the parameters $\lambda$ and $\eta$ are such that the corresponding orthogonal polynomials $P_n(x; \lambda, \eta)$ happen to belong to the Askey scheme, the evaluation of the partition function in closed form reduces to an elementary computation. As it will be shown in the following, this simple observation allows for a direct and straightforward evaluation of ASM enumerations.

## 3.3 The one-point boundary correlation function

Let us now turn to the one-point boundary correlator. Starting from its determinant representation, formula (2.8), and recalling relation (3.9), we readily write:

$$\begin{aligned} H_N^{(r)}(\lambda, \eta) = {} & \frac{(N-1)!\, \sin(2\eta)}{\big[\sin(\lambda+\eta)\big]^r \big[\sin(\lambda-\eta)\big]^{N-r+1}} \, \frac{\kappa_{N-1}(\lambda, \eta)}{h_{N-1}(\lambda, \eta)} \\ & \times P_{N-1}(\partial_\varepsilon; \lambda, \eta) \frac{(\sin\varepsilon)^{N-r} [\sin(\varepsilon - 2\eta)]^{r-1}}{[\sin(\varepsilon + \lambda - \eta)]^{N-1}} \bigg|_{\varepsilon=0}. \end{aligned} \tag{3.16}$$

This representation does not seem particularly appealing at this level, but it is worth noticing that, when the values of $\lambda$, $\eta$ are such that $P_{N-1}(x, \lambda, \eta)$ reduces to a classical polynomial, the previous correlation function may be evaluated exactly, in closed form. The use of the properties of the polynomial entering the representation is a crucial ingredient in performing such computation. This will be shown in detail later, but now we are interested instead in obtaining another equivalent representation for $H_N^{(r)}$, by making use of the crossing symmetry of the six-vertex model. The use of these two equivalent representations for the one-point boundary correlation function $H_N^{(r)}$ will then allow us to express the two-point boundary correlation function $H_N^{(r_1,r_2)}$ in terms of one-point boundary correlation functions.

We recall that the crossing symmetry is the symmetry of the vertex weights under reflection with respect to the vertical axis, and simultaneous interchange of the functions $a$ and $b$, which is equivalent to setting $\lambda \to \pi - \lambda$. Since the lattice with DWBC is invariant under the reflection with respect to the vertical axis (modulo reversal of all arrows on the horizontal edges), the crossing symmetry implies that the following relation holds

$$H_N^{(r)}(\lambda, \eta) = H_N^{(N-r+1)}(\pi - \lambda, \eta). \tag{3.17}$$

Consider expression (2.8) for the one-point function $H_N^{(r)}$. Due to (3.9) we can rewrite it as

$$H_N^{(r)}(\lambda,\eta) = \frac{(N-1)!\,\sin(2\eta)}{\big[\sin(\lambda+\eta)\big]^r\big[\sin(\lambda-\eta)\big]^{N-r+1}}\,\frac{\kappa_{N-1}(\lambda,\eta)}{h_{N-1}(\lambda,\eta)} \\ \times P_{N-1}(\partial_\varepsilon;\lambda,\eta)\frac{(\sin\varepsilon)^{N-r}[\sin(\varepsilon-2\eta)]^{r-1}}{[\sin(\varepsilon+\lambda-\eta)]^{N-1}}\bigg|_{\varepsilon=0}. \quad (3.18)$$

Taking into account (3.14) and the properties of the leading coefficient $\kappa_n(\lambda,\eta)$ and the normalization constant $h_n(\lambda,\eta)$ mentioned above, it can be easily seen that from (3.16) and (3.17) the following expression is valid as well

$$H_N^{(r)}(\lambda,\eta) = \frac{(N-1)!\,\sin(2\eta)}{\big[\sin(\lambda+\eta)\big]^r\big[\sin(\lambda-\eta)\big]^{N-r+1}}\,\frac{\kappa_{N-1}(\lambda,\eta)}{h_{N-1}(\lambda,\eta)} \\ \times P_{N-1}(\partial_\varepsilon;\lambda,\eta)\frac{(\sin\varepsilon)^{r-1}[\sin(\varepsilon+2\eta)]^{N-r}}{[\sin(\varepsilon+\lambda+\eta)]^{N-1}}\bigg|_{\varepsilon=0}. \quad (3.19)$$

Note that this expression means simply that the limit $\varepsilon\to 0$ in (3.16) can be changed into $\varepsilon\to 2\eta$ without altering the result.

Now, these two equivalent representations, (3.16) and (3.19), can be used in the study of the two-point correlation function $H_N^{(r_1,r_2)}$ given by expression (2.12), which certainly involves similar structures. Before turning to this analysis, let us put the above formulae for the one-point function in a more compact and convenient notations.

We define the functions

$$\omega(\epsilon) = \frac{\sin(\lambda+\eta)}{\sin(\lambda-\eta)}\,\frac{\sin\varepsilon}{\sin(\varepsilon-2\eta)}, \qquad \varrho(\epsilon) = \frac{\sin(\lambda-\eta)}{\sin(2\eta)}\,\frac{\sin(\varepsilon-2\eta)}{\sin(\varepsilon+\lambda-\eta)}, \quad (3.20)$$

which are related to each other as

$$\varrho(\varepsilon) = \frac{1}{\omega(\varepsilon)-1}. \quad (3.21)$$

Also we define

$$\tilde\omega(\epsilon) = \frac{\sin(\lambda-\eta)}{\sin(\lambda+\eta)}\,\frac{\sin\varepsilon}{\sin(\varepsilon+2\eta)}, \qquad \tilde\varrho(\epsilon) = \frac{\sin(\lambda+\eta)}{\sin(2\eta)}\,\frac{\sin(\varepsilon+2\eta)}{\sin(\varepsilon+\lambda+\eta)}; \quad (3.22)$$

which are in turn related to each other as

$$\tilde\varrho(\varepsilon) = \frac{1}{1-\tilde\omega(\varepsilon)}. \quad (3.23)$$

Note, that the functions with tildes are introduced such that

$$\tilde\omega(\varepsilon;\lambda,\eta) = \omega(-\varepsilon;\pi-\lambda,\eta), \qquad \tilde\varrho(\varepsilon;\lambda,\eta) = -\varrho(-\varepsilon;\pi-\lambda,\eta) \quad (3.24)$$

in accordance with the crossing symmetry considerations made above. Additionally, let us denote

$$K_{N-1}(x) = (N-1)!\, \varphi^N \frac{\kappa_{N-1}}{h_{N-1}}, P_{N-1}(x) \tag{3.25}$$

where $\varphi = \varphi(\lambda, \eta)$ is exactly the function defining entries of the matrix $\Phi$, see (2.6). In these notations formulae (3.16) and (3.19) for the correlation function $H_N^{(r)}$ read

$$H_N^{(r)} = K_{N-1}(\partial_\varepsilon) \big[\omega(\varepsilon)\big]^{N-r} [\varrho(\varepsilon)]^{N-1}\Big|_{\varepsilon=0} \tag{3.26}$$

and

$$H_N^{(r)} = K_{N-1}(\partial_\varepsilon) \big[\tilde{\omega}(\varepsilon)\big]^{r-1} [\tilde{\varrho}(\varepsilon)]^{N-1}\Big|_{\varepsilon=0}, \tag{3.27}$$

respectively.

### 3.4 The two-point boundary correlation function

Let us now consider the two-point correlation function $H_N^{(r_1,r_2)}$, which is given by formula (2.12). Obviously, function (2.13) contains all the structures introduced above apart from the factor $\sin(\varepsilon_2 - \varepsilon_1 + 2\eta)$ standing in the denominator there. However, using the identity

$$\sin(2\eta)\sin(\varepsilon_2 - \varepsilon_1 + 2\eta) = \sin\varepsilon_1 \sin\varepsilon_2 - \sin(\varepsilon_1 - 2\eta)\sin(\varepsilon_2 + 2\eta) \tag{3.28}$$

it can be easily seen that

$$\frac{\sin(\varepsilon_1 + \lambda - \eta)\sin(\varepsilon_2 + \lambda + \eta)}{\sin(\varepsilon_2 - \varepsilon_1 + 2\eta)} = \frac{1}{\varphi \varrho(\varepsilon_1)\tilde{\varrho}(\varepsilon_2)} \, \frac{1}{\omega(\varepsilon_1)\tilde{\omega}(\varepsilon_2) - 1}. \tag{3.29}$$

Thus, taking into account formula (3.11) we can write the two-point correlation function in the form

$$\begin{aligned} H_N^{(r_1,r_2)} &= [K_{N-1}(\partial_{\varepsilon_1})K_{N-2}(\partial_{\varepsilon_2}) - K_{N-2}(\partial_{\varepsilon_1})K_{N-1}(\partial_{\varepsilon_2})] \\ &\qquad \times \frac{\big[\omega(\varepsilon_1)\big]^{N-r_1} [\varrho(\varepsilon_1)]^{N-2} \big[\tilde{\omega}(\varepsilon_2)\big]^{N-r_2} [\tilde{\varrho}(\varepsilon_2)]^{N-2}}{\omega(\varepsilon_1)\tilde{\omega}(\varepsilon_2) - 1}\Bigg|_{\varepsilon_1=0,\varepsilon_2=0}. \end{aligned} \tag{3.30}$$

Taking into account that $\omega(\varepsilon), \tilde{\omega}(\varepsilon) \to 0$ as $\varepsilon \to 0$ we can expand the denominator in (3.30) in power series and it can be easily seen that only the first few terms (actually not more than $N$) of this expansion will contribute. As a result, in virtue of relations (3.21) and (3.23), we arrive to the following expression in terms of the one-point functions

$$\begin{aligned} H_N^{(r_1,r_2)} = \sum_{j=1}^{N} \Big( & H_N^{(r_1-j+1)} H_{N-1}^{(N-r_2+j)} - H_N^{(r_1-j)} H_{N-1}^{(N-r_2+j)} \\ & - H_{N-1}^{(r_1-j)} H_N^{(N-r_2+j+1)} + H_{N-1}^{(r_1-j)} H_N^{(N-r_2+j)} \Big) \end{aligned} \tag{3.31}$$

where it is assumed that if $r \leq 0$ or $r \geq N+1$ then $H_N^{(r)} = 0$ by definition.

The last expression may be equivalently expressed in terms of the generating functions, defined in (2.11), (2.15):

$$\begin{aligned} H_N(u,v) &= \frac{(u-1)H_N(u) \cdot vH_{N-1}(v) - uH_{N-1}(u) \cdot (v-1)H_N(v)}{u-v} \\ &= \frac{1}{v-u} \begin{vmatrix} uH_{N-1}(u) & vH_{N-1}(v) \\ (u-1)H_N(u) & (v-1)H_N(v) \end{vmatrix}. \end{aligned} \tag{3.32}$$

This formula generalize to arbitrary values of the vertex weights the result of paper [17] where an equivalent expression was derived in the case $\lambda = \pi/2$ and $\eta = \pi/6$, i.e., when $a = b = c$ (the so-called ice-point).

# 4 A combinatorial application

## 4.1 Alternating Sign Matrices enumerations

An alternating sign matrix (ASM) is a matrix of 1's, 0's and $-1$'s such that in each row and in each column *(i)* all nonzero entries alternate in sign, and *(ii)* the first and the last nonzero entries are 1. An example of such matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.1}$$

There are many nice results concerning ASMs, for a review, see book [9]. Many of these results have been first formulated as conjectures which were subsequently proved by different methods.

The most celebrated result concerns the total number $A(N)$ of $N \times N$ ASMs. It was conjectured in papers [24, 25] and proved in papers [15, 26] that

$$A(N) = \prod_{k=1}^{N} \frac{(3k-2)!\,(k-1)!}{(2k-1)!\,(2k-2)!} = \prod_{k=1}^{N} \frac{(3k-2)!}{(2N-k)!}. \tag{4.2}$$

A possible generalization of the previous problem consists in considering weighted enumerations, or $x$-enumerations of ASMs. In the $x$-enumeration matrices are counted with a weight $x^k$ where $k$ is the total number of '$-1$' entries in a matrix. The number of $x$-enumerated ASMs is denoted traditionally as $A(N;x)$. The extension of the $x=1$ result above to the case of generic $x$ is not known, but for a few nontrivial cases, namely $x=2$ and $x=3$ [15, 24, 25], closed expressions for $x$-enumerations are known (note that the case $x=0$ is trivial, since assigning a vanishing weight to each '$-1$' entry restricts enumeration to the sole permutation matrices: $A(n;0) = n!$).

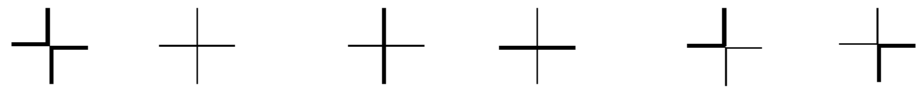



Figure 3: Vertex states—ASM's entries correspondence.

A further generalization of the previous problems consists in the so-called refined enumerations of ASMs, where one counts the number of $N \times N$ ASMs with their sole '1' of the last column at the $r$-th entry. The refined enumeration can be naturally extended to be also an $x$-enumeration. The standard notation for the refined $x$-enumeration is $A(N, r; x)$; in the case $x = 1$ one writes simply $A(N, r)$ just like $A(N)$ for the total number of ASMs. The answer for the refined $x$-enumeration is known only for the two values $x = 1$ [16, 24, 25] and $x = 2$ [25, 27].

Such enumerations can be further generalized to the doubly refined weighted countings, $A(N, r, s; x)$ where one counts the number of $N \times N$ ASMs with their sole '1' of the first and of the last column at the $r$-th and $(N - s + 1)$-th entries, respectively. An answer for $A(N, r, s; x)$ in the case $x = 1$ was found in [17].

The most direct way to derive ASM enumerations is based on the one-to-one correspondence between $N \times N$ ASMs and configurations of the six-vertex model on $N \times N$ lattice with DWBC, which has been pointed out in [27, 28], and applied for the first time in [15]. The correspondence between matrix entries and vertices is depicted in figure 3. For example, matrix (4.1) corresponds to the configuration of figure 2 and vice versa.

As an immediate consequence of this correspondence, ASM enumeration is exactly given by the partition function of square ice, when all vertex weights are set equal to unity. More generally, the number of '−1' entries in a given ASM being equal to the number of vertices of type 6 (see figure 3), and the number of vertex of type 5 and 6 being constrained by the condition $n_5 - n_6 = N$, we readily get

$$A(N; x) = (1 - x/4)^{-N^2/2}\, x^{-N/2}\, Z_N \Big|_{\substack{\lambda=\pi/2 \\ \eta=\arcsin(\sqrt{x}/2)}}. \tag{4.3}$$

Therefore, $x$-enumeration of ASM corresponds to the computation of the partition function of square ice on the subset of parameters space given by $a = b$. In this correspondence, values of $x$ belonging to the interval $(0, 4)$ corresponds to the disordered regime of the model, $-1 < \Delta < 1$.

This nice correspondence can be further extended to the refined $x$-enumeration of ASMs. In the language of square ice, the ratio $A(N, r; x)/A(N; x)$ can be rephrased as the probability of finding the unique vertex of type 5 on the first row at the $(N - r + 1)$-th site, which is exactly the definition of the boundary correlation function $H_N^{(N-r+1)}$. Explicitly, one has

$$\frac{A(N, r; x)}{A(N; x)} = H_N^{(N-r+1)} \Big|_{\substack{\lambda=\pi/2 \\ \eta=\arcsin(\sqrt{x}/2)}}. \tag{4.4}$$

Note that for the particular value $\lambda = \pi/2$, which is the one of interest in ASMs enumerations, the one-point boundary correlator enjoys the symmetry $H_N^{(r)} = H_N^{(N-r+1)}$.

Analogously, the doubly refined $x$-enumeration of ASMs is simply related to the two-point boundary correlation function $H_N^{(r,s)}$ as follows:

$$\frac{A(N,r,s;x)}{A(N;x)} = H_N^{(N-r+1,s)}\bigg|_{\substack{\lambda=\pi/2 \\ \eta=\arcsin(\sqrt{x}/2)}} . \tag{4.5}$$

It is evident from the previous formulae that 1-, 2- and 3-enumerations of ASMs correspond to the values $\eta = \pi/6$, $\pi/4$, $\pi/3$, respectively, in the six-vertex model. In these three cases, the orthogonal polynomials appearing in the representations of section 3 for the partition function and boundary correlators turn out to specialize to so-called classical ones, i.e. appearing into the Askey scheme of hypergeometric orthogonal polynomials. Such polynomials can be expressed as terminating hypergeometric series, and hence, their characteristic properties, such as orthogonality condition, three-term relation, etc., can be worked out explicitely (for full details, and $q$-analog extensions, see [19]). This hypergeometric structure allows one to evaluate ASM enumerations in a simple and explicit way [23].

## 4.2 1-enumerations of ASMs

To illustrate the method we shall here restrict ourselves to the case of 1-enumerations of ASMs. This corresponds to the so-called ice-point, or $\Delta = 1/2$ symmetric point, of the six-vertex model. The parameters of the model are specialized to $\eta = \pi/6$ and $\lambda = \pi/2$, and correspondingly, the Boltzmann weights assume the values $a = b = c = 1$. The orthogonality weight reads

$$\mu(x) = \frac{\sinh \frac{\pi}{6} x}{\sinh \frac{\pi}{2} x} = \frac{1}{4\pi^2} \left| \Gamma\left(\frac{1}{3} + i\frac{x}{6}\right) \Gamma\left(\frac{2}{3} + i\frac{x}{6}\right) \right|^2 . \tag{4.6}$$

Direct inspection of some tables of classical orthogonal polynomials, such as [19], allows to recognize here a particular specialization of the orthogonality weight for Continuous Hahn polynomials, which are defined as

$$p_n(x;a,b,c,d) = \mathrm{i}^n \frac{(a+c)_n (a+d)_n}{n!}\, {}_3F_2\left( \left. \begin{matrix} -n\,,\ n+a+b+c+d-1\,,\ a+\mathrm{i}x \\ a+c\,,\ a+d \end{matrix} \right| 1 \right), \tag{4.7}$$

and satisfy the orthogonality relation

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} p_n(x;a,b,c,d) p_m(x;a,b,c,d)\, \Gamma(a+\mathrm{i}x)\, \Gamma(b+\mathrm{i}x)\, \Gamma(c-\mathrm{i}x)\, \Gamma(d-\mathrm{i}x)\, \mathrm{d}x \\ = \frac{\Gamma(n+a+c)\, \Gamma(n+a+d)\, \Gamma(n+b+c)\, \Gamma(n+b+d)}{(2n+a+b+c+d-1)\, \Gamma(n+a+b+c+d-1)\, n!}\, \delta_{nm} . \end{aligned} \tag{4.8}$$

Orthogonality condition (4.8) is valid if the parameters $a,b,c,d$ satisfy $\mathrm{Re}\,(a,b,c,d)>0$, $a=\bar{c}$ and $b=\bar{d}$. Comparing (4.6) with (4.8) we are naturally led to the choice of parameters $a=c=1/3$ and $b=d=2/3$. Hence, the appropriate polynomials to be associated to the Hankel determinant in this case are

$$P_n(x)=p_n\left(\frac{x}{6};\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3}\right)=\mathrm{i}^n(2/3)_n\,{}_3F_2\left(\left.\begin{matrix}-n\,,\ n+1\,,\ 1/3+\mathrm{i}x/6\\ 2/3\,,\ 1\end{matrix}\right|1\right). \tag{4.9}$$

The normalization constant and the leading coefficient are readily computed:

$$h_n=\frac{2(3n+1)!}{(2n+1)\,3^{3n+1/2}\,n!},\qquad \kappa_n=\frac{(2n)!}{6^n\,(n!)^2} \tag{4.10}$$

Substituting the obtained values of $h_n$ and $\kappa_n$ in expression (3.15) for the partition function, and cancelling whatever possible, we arrive to the following value for the ice point partition function

$$Z_N\Big|_{\substack{\lambda=\pi/2\\ \eta=\pi/6}}=\left(\frac{\sqrt{3}}{2}\right)^{N^2}\prod_{n=0}^{N-1}\frac{(3n+1)!\,n!}{(2n)!\,(2n+1)!}. \tag{4.11}$$

The product expression here gives exactly the total number of ASMs, $A(N)$, since by formula (4.3) the first factor relates $A(N)$ with the partition function,

$$A(N)=(3/4)^{-N^2/2}\,Z_N\Big|_{\substack{\lambda=\pi/2\\ \eta=\pi/6}}. \tag{4.12}$$

Thus, we have easily recovered the celebrated result, Eqn. (4.2), for 1-enumeration of ASMs. It is worth noting that the proof presented here is considerably simpler in comparison to those of papers [15, 26].

The refined enumeration $A(N,r)$ can be obtained within the same framework, the derivation being only slightly more involved. The key ingredient is now that the orthogonal polynomials belonging to the Askey scheme are known to satisfy some differential or finite difference equation in their variable. In the present case, such equation reads

$$\begin{aligned}\left(\frac{1}{3}-\frac{\mathrm{i}x}{6}\right)\left(\frac{2}{3}-\frac{\mathrm{i}x}{6}\right)P_{N-1}(x+6\mathrm{i})+\left[\frac{x^2}{18}-\frac{4}{9}-N(N-1)\right]P_{N-1}(x)\\ +\left(\frac{1}{3}+\frac{\mathrm{i}x}{6}\right)\left(\frac{2}{3}+\frac{\mathrm{i}x}{6}\right)P_{N-1}(x-6\mathrm{i})=0\,.\end{aligned} \tag{4.13}$$

First we recall that the refined enumeration is, modulo an obvious overall normalization factor, see (4.4), nothing but the one-point boundary correlator, which in the present case ($\eta=\pi/6$ and $\lambda=\pi/2$) reads:

$$H_N^{(r)}=H_N^{(N-r+1)}=\mathrm{const}\times\left\{P_{N-1}(\partial_\varepsilon)[\varrho(\varepsilon)]^{N-1}[\omega(\epsilon)]^{r-1}\right\}\Big|_{\varepsilon=0} \tag{4.14}$$

where $\omega(\varepsilon)$ and $\varrho(\varepsilon)$ are given by

$$\omega(\epsilon) = \frac{\sin\varepsilon}{\sin(\varepsilon - \pi/3)}, \qquad \varrho(\epsilon) = \frac{\sin(\varepsilon - \pi/3)}{\sin(\varepsilon + \pi/3)}. \tag{4.15}$$

Then, exploiting the simple relation $P_{N-1}(\partial_\varepsilon \pm 6\mathrm{i}) = \mathrm{e}^{\mp 6\mathrm{i}\varepsilon} P_{N-1}(\partial_\varepsilon)\mathrm{e}^{\pm 6\mathrm{i}\varepsilon}$, it is easy to see that finite difference equation (4.13) implies the condition

$$P_{N-1}(\partial_\varepsilon)\ \left[\sin 3\varepsilon\, \partial_\varepsilon^2\, \sin 3\varepsilon + \sin^2 3\varepsilon - 9N(N-1)\right]\ [\varrho(\epsilon)]^{N-1}[\omega(\epsilon)]^{r-1}\tau(\epsilon)\Big|_{\varepsilon=0} = 0 \tag{4.16}$$

where $\tau(\varepsilon)$ is at this stage an arbitrary function. Our aim now is to determine the form of $\tau(\varepsilon)$ in such a way that the last equation can be turned into a recurrence relation in $r$ for the boundary correlator $H_N^{(r)}$. A constructive procedure has been devised in [23] to perform this task in general. The derivation is given there in full detail. In the present case the resulting recurrence relation reads

$$r(r-2N+1)H_N^{(r+1)} - (r-N)(N+r-1)H_N^{(r)} = 0. \tag{4.17}$$

This recurrence can be easily solved modulo a normalization constant

$$H_N^{(r)} = \text{const} \times \frac{(N+r-2)!\,(2N-1-r)!}{(r-1)!\,(N-r)!}. \tag{4.18}$$

A possible way to fit the normalization condition (2.10), is to consider the generating function $H_N(z)$ defined via Eqn. (2.11). The result reads

$$H_N(z) = \frac{(2N-1)!\,(2N-2)!}{(N-1)!\,(3N-2)!}\ {}_2F_1\left(\begin{matrix}1-N\,,\ N\\ 2-2N\end{matrix}\middle|\, z\right) \tag{4.19}$$

where the proper normalization is easily determined through Chu-Vandermonde identity

$${}_2F_1\left(\begin{matrix}-m\,,\ b\\ c\end{matrix}\middle|\, 1\right) = \frac{(c-b)_m}{(c)_m}; \qquad (a)_m := a(a+1)\cdots(a+m-1). \tag{4.20}$$

Inspecting the coefficient of $z^{r-1}$ in (4.19) one obtains

$$H_N^{(r)} = \frac{\binom{N+r-2}{N-1}\binom{2N-1-r}{N-1}}{\binom{3N-2}{N-1}}. \tag{4.21}$$

The refined 1-enumeration of ASMs, conjectured in [25], and first proven in [16], immediately follows from relation (4.4).

Let us conclude with the doubly refined 1-enumeration of ASMs, which is readily obtained by substituting expression (4.21) for the refined enumeration into (3.31). Analogously, the corresponding generating function, $H_N(u,v)$, is immediately obtained by substituting (4.19) into (3.32), hence recovering the result of paper [17] where a different approach was considered.

## 4.3 Other ASMs enumerations

We have just seen that the present approach allows a very simple, straightforward and unified derivation of all answers concerning 1-enumerations of ASMs. These answers were already known, but the method exposed above can be extended to all $x$-enumerations where the underlying orthogonal polynomials appear to belong to the Askey scheme. Indeed, when this is the case, $h_n$, $\kappa_n$ are explicitly known, and moreover the considered orthogonal polynomials happen to satisfy a finite difference equation in their variable; has illustrated above, this is sufficient to compute explicitly the ordinary, refined, and doubly refined enumerations. As a matter of fact, it appears that the underlying orthogonal polynomials belong to the Askey scheme only for $x = 1, 2, 3$ (for completeness we mention also the trivial case $x = 0$, where Meixner-Pollaczek orthogonality measure naturally emerges when the $\eta = 0$ limit is performed in (3.13), and the highly non-trivial case $x = 4$, where the corresponding orthogonal polynomials turns out to be of Bannai-Ito type [30], and therefore do not fall within Askey scheme). Our approach allows to recover in a straightforward way the results concerning 2-enumerations, and to derive the previously unknown answers for the refined and doubly refined 3-enumerations.

The 2-enumerations correspond to the so-called free-fermion point of the square ice, where the parameters of the model assume the values $\lambda = \pi/2$, $\eta = \pi/4$. As a matter of fact, the whole free-fermion line of the model ($\Delta = 0$) can be treated at once, with the crossing parameter set to the value $\eta = \pi/4$, while $\lambda$ is free to vary in the interval $\pi/4 < \lambda < 3\pi/4$. In this case the associated orthogonal polynomials can be recognized as a particular specialization of Meixner-Pollaczek polynomials [19]. The computation of the partition function and boundary correlators, and thus of the various ASM 2-enumerations, can be performed along the lines of the proposed approach [23], and is particularly simple. The various answers in this case have been known for some time [4, 5, 25, 27].

Let us now discuss the 3-enumerations, which correspond to the so-called dual ice-point ($\Delta = -1/2$) of the six-vertex model, with $\lambda = \pi/2$ and $\eta = \pi/3$. In this case one can recognize that the measure is associated to the Continuous Dual Hahn polynomials [19]. A complication however occurs: on one hand these are polynomials in $x^2$ rather than in $x$, and on the other hand the support of the integration measure is restricted to the positive half-axis. These inconveniences are however easily circumvented, provided that the cases of even and odd $N$ ($N$ being the size of the lattice, or of the ASMs) are treated separately. Indeed the set of polynomials $P_n(x)$ associated to the determinant representation for the partition function or the boundary correlation functions should be specified differently for even or odd $n$, each of these two cases corresponding to a slightly different choice of parameters in the Continuous Dual Hahn polynomials. This is in fact a merely technical complication and the general procedure previously outlined may again be applied without modification. This has been done in full detail in [23]. We shall here just recall the main results. The known answer for the partition function, or, equivalently, for the ordinary 3-enumeration of ASMs [15, 25], is readily recovered. Moreover the previously unknown answer for the one-point boundary correlator (or, equivalently, for the refined

3-enumeration of ASMs) can be worked out, and reads:

$$
\begin{aligned}
H^{(r)}_{2m+2} &= \frac{B(m,r-1)+B(m,r-2)}{2} \\
H^{(r)}_{2m+3} &= \frac{2\,B(m,r-1)+5\,B(m,r-2)+2\,B(m,r-3)}{9}
\end{aligned}
\tag{4.22}
$$

where the quantities $B(m,n)$, obeying $B(m,n)=B(m,2m-n)$, are given by

$$
\begin{aligned}
B(m,n) = \frac{(2m+1)!\,m!}{3^m\,(3m+2)!} \sum_{\ell=\max(0,n-m)}^{[n/2]} & (2m+2-n+2\ell)\binom{3m+3}{n-2\ell} \\
& \times \binom{2m+\ell-n+1}{m+1}\binom{m+\ell+1}{m+1} 2^{n-2\ell}
\end{aligned}
\tag{4.23}
$$

for $n=0,1,\ldots,2m$, while they are assumed to vanish otherwise. It should be mentioned that this result has first been obtained in [31] through the solution of a second order differential equation associated to the generating function $H_N(u)$, see (2.11). Such differential equation, which on one hand is equivalent to the recurrence relation which can be derived from the orthogonal polynomial approach (i.e. the analogue of (4.17) for the present case), is on the other hand closely related to the functional equation proposed in [32], within a different approach to the problem.

The two-point boundary correlator is immediately obtained simply by inserting the previous expression into (3.31).

# 5 An alternative representation for the (partially) inhomogeneous partition function

We have till now restricted ourselves to the homogeneous version of the six-vertex model. This has allowed us to apply very standard tools from the theory of orthogonal polynomials to derive in a simplified and unified way several nontrivial results. Of course, in doing so, we have lost the dependence in, let us say, the partition function, on the spectral parameters of the model. This dependence implicitly encodes information about the correlation functions of the model, and is thus of great interest. We want here to show how the results of section 3 for the boundary correlators, based on the proposed orthogonal polynomial representation, can be used to recover the dependence of the partition function on the spectral parameters of the (partially) inhomogeneous version of the six-vertex model.

As briefly explained in section 2, when considering the inhomogeneous version of the six-vertex model, one associates a spectral parameter $\lambda_\alpha$ ($\alpha=1,\ldots,N$) to each row of the lattice, and $\nu_\beta$ ($\beta=1,\ldots,N$) to each column. We want here to consider a particular situation in which all horizontal inhomogeneities $\nu_\beta$ are set to zero, and a

subset of the vertical one is set to some value $\lambda$. For simplicity, and comparison with previous works [17, 18] we shall fix this value to $\lambda = \pi/2$ (note however that our results can be readily extended to the case of generic values of $\lambda$). On the other hand $\eta$ will be considered generic. We reparametrize the Boltzmann weights (2.4) as follows:

$$a(z) = zq + 1\,, \qquad b(z) = z + q\,, \qquad c(z) = (1-q^2)\sqrt{-\frac{z}{q}}\,, \tag{5.1}$$

where

$$z_\alpha = \mathrm{e}^{2i(\lambda_\alpha - \pi/2)}\,, \qquad q = \mathrm{e}^{2i\eta}\,. \tag{5.2}$$

We have here ignored the parameters $\nu_\alpha$, which have all been set to zero; the definition of the Boltzmann weights in (5.1) moreover differs from (2.4) by a common overall factor. This choice turns out to be more convenient for what follows, and can be absorbed in the normalization of the partition function, which is now a function of $N$ variables: $Z_N(z_1, \dots, z_N)$.

In the case when only one inhomogeneity is present, in the first column, the partition function can be expressed in terms of the boundary correlator $H_N^{(r)}$ as follows:

$$\begin{aligned} Z_N(z, 1, \dots, 1) &= Z_N(1, 1, \dots, 1) \sum_{r=1}^{N} \left(\frac{b(z)}{b(1)}\right)^{r-1} \frac{c(z)}{c(1)} \left(\frac{a(z)}{a(1)}\right)^{N-r} H_N^{(r)} \\ &= Z_N(1, 1, \dots, 1)\, \frac{c(z)}{c(1)} \left(\frac{a(z)}{a(1)}\right)^{N-1} H_N\big(b(z)/a(z)\big)\,, \end{aligned} \tag{5.3}$$

where the generating function $H_N(w)$, see (2.11), is a polynomial of order $N-1$ in $w$. Let us now introduce the variable $u$, related to $z$ as follows

$$u = \frac{b(z)}{a(z)} = \frac{z+q}{1+zq}\,, \qquad z = \frac{u-q}{1-uq}\,, \tag{5.4}$$

and consider the partition function as a function of the variables $u_1, \dots, u_n$, by introducing

$$\tilde{Z}_N(u_1, \dots, u_N) = \frac{Z_N(z_1, \dots, z_N)}{Z_N(1, \dots, 1)} \prod_{\alpha=1}^{N} \left[\frac{c(1)}{c(z_\alpha)} \left(\frac{a(1)}{a(z_\alpha)}\right)^{N-1}\right]. \tag{5.5}$$

Note that the 'partition function' $\tilde{Z}_N(u_1, \dots, u_N)$ is normalized such that $\tilde{Z}_N(1, \dots, 1) = 1$. Relation (5.3) then reads

$$\tilde{Z}_N\,(u, 1, \dots, 1) = H_N(u)\,. \tag{5.6}$$

In the same way, following [17, 18], we may rewrite the partition function with two inhomogeneities as the generating function (2.15) of the two-point boundary correlator. Since the latter is in turn expressible in terms one-point boundary correlators, see (3.32), we immediately have

$$\tilde{Z}_N\,(u_1, u_2, 1, \dots, 1) = \frac{1}{\Delta(u_1, u_2)} \begin{vmatrix} u_1 H_{N-1}(u_1) & u_2 H_{N-1}(u_2) \\ (u_1 - 1) H_N(u_1) & (u_2 - 1) H_N(u_2) \end{vmatrix}. \tag{5.7}$$

It is now quite natural to guess that

$$\tilde{Z}_N\left(u_1,\ldots,u_k,1,\ldots,1\right)=\frac{1}{\Delta(u_1,\ldots,u_k)}$$
$$\times\begin{vmatrix} u_1^{k-1}H_{N-k+1}(u_1) & u_2^{k-1}H_{N-k+1}(u_2) & \ldots & u_k^{k-1}H_{N-k+1}(u_k) \\ u_1^{k-2}(u_1-1)H_{N-k+2}(u_1) & u_2^{k-2}(u_2-1)H_{N-k+2}(u_2) & \ldots & u_k^{k-2}(u_k-1)H_{N-k+2}(u_k) \\ \cdots & \cdots & \cdots & \cdots \\ (u_1-1)^{k-1}H_N(u_1) & (u_2-1)^{k-1}H_N(u_2) & \ldots & (u_k-1)^{k-1}H_N(u_k) \end{vmatrix}. \tag{5.8}$$

We note that the right hand side is by construction a symmetric polynomial in $u_1,\ldots,u_k$, of order $N-1$ in each variable, as it should. Moreover it is evident that the homogeneous limit can be performed one variable at a time, say $u_k\to 1$, maintaining the proposed structure: formula (5.8) obviously satisfies

$$\lim_{u_k\to 1}\tilde{Z}_N\left(u_1,\ldots,u_{k-1},u_k,1,\ldots,1\right)=\tilde{Z}_N\left(u_1,\ldots,u_{k-1},1,1,\ldots,1\right). \tag{5.9}$$

It is also straightforward to verify that

$$\lim_{u_k\to 0}\tilde{Z}_N\left(u_1,\ldots,u_{k-1},u_k,1,\ldots,1\right)=H_N(0)\,\tilde{Z}_{N-1}\left(u_1,\ldots,u_{k-1},1,\ldots,1\right), \tag{5.10}$$

which is nothing but Korepin recursion relation [1] specialized to the present situation. It should however be mentioned that in the present case, where all 'horizontal' spectral parameters $\nu_\beta$ have been set to same value, the uniqueness of the solution of such a recursion relation is not guaranteed. A complete derivation of representation (5.8) can however be given by exploiting a set of identities relating the derivatives of function $\varphi(\lambda,\eta)$ to the polynomials $H_N(u)$; this will be done elsewhere.

We would like to conclude by emphasizing that representation (5.8) constitutes a rather wide generalization of previous expressions given in [17, 18]. First of all it generalizes the previous formulae, holding in the case of two spectral parameters, to a larger number of variables. Moreover, the proposed representation holds for any values of $\eta$, (and can be extended straightforwardly to any value of $\lambda\neq\pi/2$, by a suitable redefinition of variable $u$), expressing in general the partition function in terms of just one-point boundary correlators of the corresponding homogeneous model.

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